Level curves, crossings and specular points for Gaussian models.

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Level curves, crossings and speed of specular points for Gaussian models

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Abstract

Using generalizations of the well known Rice formula and applying the general method related to m-dependent processes that we settled in earlier works, allow one to obtain representations into the Itô-Wiener Chaos and CLTs for curve-crossings number. This approach not only explains heuristic considerations of Longuet-Higgins on specular points and related problems in the context of sea modelling, but goes far beyond when providing asymptotic results. These results on curve-crossings may also be applied in other fields. One example is the study of the estimator of the natural frequency of a harmonic oscillator.

1 Introduction

Recently there has been a renewed interest in applying the generalizations of the Rice formulae to explain some difficult phenomena in optics (see for instance [4]). In this work, we also use generalizations of the well known Rice formula (see [21]) and apply the general method we settled in earlier works (see [12], [14], or [16] for a survey on results and methods) to obtain representations into the Itô-Wiener Chaos and CLTs for level functionals of stationary Gaussian processes. It allows one to provide results on curve-crossings and specular points, which may explain for instance the light reflection on the sea surface.
Note that this last subject was heuristically developed in the fifties and sixties by Longuet-Higgins (see [18] and [19]), whose main motivation was to determine the height of the waves.
The approach we chose allows us to explain the Longuet-Higgins discovery by proving his heuristic formulae, and goes far beyond, providing variance results and CLTs, which may apply in various applied research areas, as e.g. the ones considered here: sea modelling and random oscillation.
Let us also mention related works, such as Piterbarg and Rychlik's CLT for wave functionals (see [20]) or the recent paper from Lindgren (see [17]) who introduced a Gaussian Lagrangian model to describe the sea surface.
The paper is organized as follows. In section 2, we develop the expansion into Hermite polynomials for the number of crossings of a differentiable curve, generalizing an earlier result we obtained for a fixed level (see [12]). Then we describe the asymptotic behavior of this expansion according to the form of the level curve, namely if the curve is periodic or linear. The proofs of those results are given in the Appendix. As an application when the curve is periodic, we study the estimator of the natural frequency of a harmonic oscillator with periodic forcing term, whereas the case of linear curves is applied to the study of specular points. Section 3 is devoted to the speed of specular points on the sea surface.

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2 Asymptotic behavior of the curve-crossings number

Hermite polynomial expansion (or Multiple Wiener-Itô Integrals) may be a powerful tool to represent and to study nonlinear functionals of stationary Gaussian processes. Thus we will first provide the Hermite expansion for the curve-crossings number in order to obtain in an easier way its asymptotic behavior, which depends on the curve's type.

$(H_n)_{n\geq 0}$ will denote the Hermite polynomials, defined by $H_n(x) = (-1)^n e^{x^2/2}\dfrac{d^n}{dx^n}\, e^{-x^2/2}$, which constitute a complete orthogonal system in the Hilbert space $L^2(I\!R, \varphi(u)du)$, φ being the standard normal density.

2.1 Hermite expansion for the curve-crossings number

Let $X = \{X_t, t \in I\!R\}$ be a centered stationary Gaussian process, variance one, with twice differentiable correlation function r given by $r(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda}F(d\lambda)$, where F is the spectral measure.

Let $N_t^X(\psi)$ be the number of crossings of a differentiable function ψ by the process X :

$$N_t^X(\psi) = card\{s \leq t : X_s = \psi_s\}.$$

The random variable $N_t^X(\psi)$ can also be seen as the number $N_t^Y(0)$ of zero crossings by the non-stationary (but stationary in the sense of the covariance) Gaussian process $Y = \{Y_s,\ s \in I\!R\}$ defined by $Y_s := X_s - \psi_s$, i.e. $N_t^X(\psi) = N_t^Y(0)$.

Suppose that r satisfies on $[0, \delta]$, $\delta > 0$,

$$r(\tau) \;=\; 1 + \frac{r''(0)}{2}\tau^2 + \theta(\tau), \quad \text{with } \theta(\tau) > 0,\ \frac{\theta(\tau)}{\tau^2} \to 0,\ \frac{\theta'(\tau)}{\tau} \to 0,\ \theta''(\tau) \to 0,\ \text{as } \tau \to 0, \qquad (1)$$

that the nonnegative function L defined by $L(\tau) := \dfrac{\theta''(\tau)}{\tau} = \dfrac{r''(\tau) - r''(0)}{\tau}$, $\tau > 0$, satisfies the Geman condition:

$$\exists \delta > 0, \quad L \in L^1([0, \delta]) \qquad (2)$$

and assume that the modulus of continuity of $\dot\psi$ defined by $\gamma(\tau) := \sup\limits_{u\in[0,t]} \sup\limits_{|s|\leq\tau} |\dot\psi(u + s) - \dot\psi(u)|$ is such that

$$\int_0^\delta \frac{\gamma(s)}{s}\, ds < \infty. \qquad (3)$$

These conditions imply that $N_t^X(\psi)$ has a finite variance, as it has been proved by the authors in [15]. Notice that we still do not know whether, under the Geman condition, $\int_0^\delta \frac{\gamma(s)}{s}\, ds = \infty$ implies that $N_t^X(\psi)$ does not belong to $L^2(\Omega)$.

Let $a_l(m)$ be the coefficients in the Hermite's basis of the function $|\ .-m|$, given by

$$\begin{aligned} a_0(m) &= \mathbb{E}|Z - m|, \quad \text{where } Z \text{ is a standard Gaussian r.v.,} \\ &= m\,[2\Phi(m) - 1] + \sqrt{\frac{2}{\pi}}\, e^{-\frac{m^2}{2}} = \sqrt{\frac{2}{\pi}}\left[1 + \int_0^m \int_0^u e^{-\frac{v^2}{2}} dv du\right], \\ \text{and} \qquad a_l(m) &= (-1)^{l+1}\sqrt{\frac{2}{\pi}}\frac{1}{l!}\int_0^1 e^{-\frac{m^2y^2}{2}} H_l(-my)y^{l-2}dy, \qquad l \geq 1. \end{aligned} \qquad (4)$$

The following proposition gives the Hermite expansion for the curve-crossings number under the only hypothesis that the number of crossings belongs to $L^2(\Omega)$, thus completes the results of [12] for a fixed level $\psi_t \equiv x$ and for many functions ψ.

Proposition 1 *Under the conditions (1), (2) and (3), the number of crossings $N_t^X(\psi)$ of the function ψ by the process X has the following expansion in $L^2(\Omega)$:*

$$N_t^X(\psi) = \sqrt{-r''(0)}\, \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^{q} \frac{H_{q-l}(\psi_s)\varphi(\psi_s)}{(q-l)!}\, a_l\left(\frac{\dot\psi_s}{\sqrt{-r''(0)}}\right) H_{q-l}(X_s) H_l\left(\frac{\dot X_s}{\sqrt{-r''(0)}}\right) ds, \qquad (5)$$

$a_l(.)$ being defined in (4).
This result can be easily generalized to a process X satisfying $\mathbb{E}[X_t^2] = r(0)$:

$$N_t^X(\psi) = \sqrt{\frac{-r''(0)}{r(0)}} \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^{q} \kappa_{ql}(s) H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right) ds, \tag{6}$$

$$\textit{where} \quad \kappa_{ql}(s) \ := \ \frac{H_{q-l}\left(\frac{\psi_s}{\sqrt{r(0)}}\right)\varphi\left(\frac{\psi_s}{\sqrt{r(0)}}\right)}{(q-l)!} a_l\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right).$$

We generalized the approach settled in [12] to obtain such a result considered for the first time by Slud in [22]. Whereas this author provided a MWI expansion of $N_t^X(\psi)$ by approximating the crossings of the process X by those of the discrete version of X, our method consists in approaching the process, then the crossings, by using a process smoothed-by-convolution, for which the expansion can be readily obtained. The main ideas of the proof, which uses also technical tools developed in [15], can be found in the Appendix.

2.2 Asymptotic behavior according to the level curve's type

We shall describe the asymptotic behavior of the previous expansion of the number of curve crossings according to the form of the curve ψ.

> If the function ψ is periodic, then a CLT can be deduced since we may say that we remain in an ergodic situation similar to the fixed barrier problem. As an example, let us mention the case of a cosine barrier ψ of the type $\psi(x) = \sqrt{2A}\cos(\omega x)$, first studied by Rice, then by Cramér & Leadbetter (see [9]). We can also mention the case of a harmonic oscillator driven by a white noise and with periodic forcing term, that will be developed below.
>
> If ψ is a linear function of the time, i.e. $\psi_s := ks$, we shall prove that the number of ψ-crossings of X belongs to $L^2(\Omega)$ as the time tends to infinity. This case encounters the number of specular points of a curve, that will be studied in the next section.

The proofs of both theorems rely mainly on the method developed in [14] and are given in the Appendix.

2.2.1 Periodic curve

Suppose that the function ψ is p-periodic, *i.e.* $\psi_{s+p} = \psi_s$. So is the function κ_{ql} defined in Proposition 1, for all q and l.
We are interested in the asymptotic behavior of the r.v. $N_t^X(\psi) - \mathbb{E}[N_t^X(\psi)]$ as $t \to \infty$.
Let us introduce (see [1])

$$\chi(s) := \sup\left(|r(s)| + \frac{|r'(s)|}{\sqrt{-r''(0)}}, \frac{|r'(s)|}{\sqrt{-r''(0)}} + \frac{|r''(s)|}{-r''(0)}\right). \tag{7}$$

Theorem 1 *:* **CLT for** $N_t^X(\psi)$.
Let X and ψ both satisfying the hypothesis of Proposition 1. If

$$\int_0^\infty \chi(s) ds < \infty \tag{8}$$

then

$$\frac{N_{np}^X(\psi) - \mathbb{E}[N_{np}^X(\psi)]}{\sqrt{n}} \quad \xrightarrow[n\to\infty]{d} \quad \mathcal{N}\left(0, \sum_{q=1}^{\infty} \sigma^2(q)\right),$$

where $\sigma^2(q)$ is defined by

$$\begin{aligned}\sigma^2(q) &= \int_0^\infty \sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\left(\int_0^p \kappa_{ql_1}(u)\kappa_{ql_2}(u-z)du\right)\\ &\quad \mathbb{E}\left[H_{q-l_1}\left(\frac{X_z}{\sqrt{r(0)}}\right)H_{l_1}\left(\frac{\dot{X}_z}{\sqrt{-r''(0)}}\right)H_{q-l_2}\left(\frac{X_0}{\sqrt{r(0)}}\right)H_{l_2}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]dz.\end{aligned}$$

The same result holds when replacing np by t.

Application to a harmonic oscillator.

The following application aims at clarifying, from a mathematical point of view, some concepts linked to the synchronization phenomenon, in particular to the phase synchronization. To study such a phenomenon, it is necessary to compare the average phase velocity between different signals. This quantity turns out to be the average rate of zero crossings by the signals and can thus be obtained by using Rice's formula. In the physics literature, this average phase velocity is called the *Rice frequency* and is usually denoted by $<\omega>_R$.
When considering a harmonic oscillator driven by a white noise, the average of zero crossings provides in stationary regime the natural frequency ω_0 of the oscillator, as we will see below. If, moreover, the oscillator is driven by an additional deterministic periodic signal, another behavior is observed. In [8] an interesting discussion emphasizes the behavior of the Rice frequency depending on the existence and intensity of a noise. Recall that in the case of no noise, the system behaves with the same frequency as the one of the periodic driven force.

Let a harmonic oscillator $X=(X_t)$ driven by a Gaussian white noise be the stationary solution of the stochastic differential equation

$$\ddot{X}_t+\gamma\dot{X}_t+\omega_0^2X_t=\sigma dW_t, \tag{9}$$

where the parameters $\gamma>0$, ω_0^2, σ^2 are, respectively, the damping factor, the natural frequency of the oscillator, the noise's variance, and W is a standard Gaussian white noise.
Suppose that the system is underdamped, i.e. $2\omega_0>\gamma$.
The process X is Gaussian, zero mean and with spectral density

$$f(\lambda)=\frac{\sigma^2}{2\pi[(\lambda^2-\omega_0^2)^2+\gamma^2\lambda^2]}, \quad \lambda\in \mathbb{R}.$$

Computing the spectral moments, we obtain

$$m_0:=r(0)=2\int_0^\infty f(\lambda)d\lambda=\frac{\sigma^2}{2\gamma\omega_0^2} \quad \text{and} \quad m_2:=-r''(0)=2\int_0^\infty \lambda^2 f(\lambda)d\lambda=\frac{\sigma^2}{2\gamma}.$$

The Ergodic Theorem and the Rice formula imply that

$$\lim_{t\to\infty}\frac{N_t^X(0)}{t}=\frac{1}{\pi}\sqrt{\frac{m_2}{m_0}}=\frac{\omega_0}{\pi} \quad \text{a.s.},$$

from which can be deduced an a.s. consistent estimator of the natural frequency ω_0 of the oscillator or of the Rice frequency $<\omega>_R$ since it is defined in terms of the zero crossings number $N_t^X(0)$ of X by $<\omega>_R=\lim\limits_{t\to\infty}\dfrac{N_t^X(0)}{t}$, so $\omega_0=\pi<\omega>_R$.
Let us prove the asymptotical normality of the estimator of ω_0.
First we check that $\mathbb{E}[N_t^X(0)]^2<\infty$. To do so, we prove that the covariance function satisfies the Geman

condition.
Indeed, we have

$$\int_0^\delta \frac{r''(t)-r''(0)}{t}\,dt \;=\; \frac{\sigma^2}{\pi}\int_0^\infty \left(\int_0^\delta \frac{1-\cos(\lambda t)}{t}\,dt\right)\frac{\lambda^2}{(\lambda^2-\omega_0^2)^2+\gamma^2\lambda^2}\,d\lambda.$$

But

$$\int_0^\delta \frac{1-\cos(\lambda t)}{t}\,dt \;=\; 2\int_0^{\lambda\delta}\frac{\sin^2(2u)}{u}\,du \;=\; 2\left(\int_0^a \frac{\sin^2(2u)}{u}\,du \;+\; \int_a^{\lambda\delta}\frac{\sin^2(2u)}{u}\,du\right) \;\le\; C\left(a^2/2 \;+\; |\log(\lambda\delta)|\right),$$

with C some positive constant which may vary from line to line,
so that $\int_0^\delta \frac{1-\cos(\lambda t)}{t}\,dt \;\le\; C|\log(\lambda\delta)|$, hence

$$\int_0^\delta \frac{r''(t)-r''(0)}{t}\,dt \;\le\; C\frac{\sigma^2}{\pi}\int_0^\infty |\log(\lambda\delta)|\frac{\lambda^2}{[(\lambda^2-\omega_0^2)^2+\gamma^2\lambda^2]}\,d\lambda<\infty.$$

Now we can apply Proposition 1 to obtain

$$\sqrt{t}\left(\frac{N_t^X(0)}{t}-\frac{\omega_0}{\pi}\right)=\frac{\omega_0}{\pi}\sum_{q=1}^{\infty}\sum_{l=0}^{[q/2]} b_{2(q-l)}\tilde{a}_{2l}\frac{1}{\sqrt{t}}\int_0^t H_{2(q-l)}(X_s/\sqrt{m_0})H_{2l}(\dot{X}_s/\sqrt{m_2})ds,$$

where $b_{2k}=\frac{H_{2k}(0)}{(2k)!}$ and $\tilde{a}_{2l}=\sqrt{\frac{\pi}{2}}\,a_{2l}(0)=\frac{(-1)^{l+1}}{2^l l!(2l-1)}$.
Then, since (8) is satisfied because the covariance function tends exponentially towards zero as $t\to\infty$, Theorem 1 entails:

$$\sqrt{t}\left(\frac{N_t^X(0)}{t}-\frac{1}{\pi}\,\omega_0\right)\;\underset{n\to\infty}{\overset{d}{\longrightarrow}}\;\mathcal{N}(0,\;\omega_0^2\Sigma^2/\pi^2),\qquad \textit{where}\quad \Sigma^2=\sum_{q=1}^{\infty}\sigma^2(q)$$

$$\begin{aligned}\textit{with}\quad \sigma^2(q)\;&=\;\sum_{l_1=0}^{[q/2]}\sum_{l_2=0}^{[q/2]} b_{2(q-l_1)}\tilde{a}_{2l_1}b_{2(q-l_2)}\tilde{a}_{2l_2}\\ &\quad\times\int_0^\infty \mathbb{E}\left[H_{2(q-l_1)}\left(\frac{X_0}{\sqrt{m_0}}\right)H_{2l_1}\left(\frac{\dot{X}_0}{\sqrt{m_2}}\right)H_{2(q-l_1)}\left(\frac{X_s}{\sqrt{m_0}}\right)H_{2l_1}\left(\frac{\dot{X}_s}{\sqrt{m_2}}\right)\right]ds.\end{aligned}$$

Note that this result could be used to build bootstrapping confidence intervals for ω_0.

Case of a forcing linear harmonic oscillator.
Our result on the crossings of a periodic curve allows us to consider another interesting situation, namely that of a forcing linear harmonic oscillator driven by a Gaussian white noise.
Consider the process solution of the equation

$$\ddot{Y}_t+\gamma\dot{Y}_t+\omega_0^2Y_t=\sigma dW_t+F\cos(\alpha t),$$

where $f(t)=F\cos(\alpha t)$ is the forcing function, 2π-periodic, with $\alpha>0$ and $F>0$.
As $t\to\infty$, the solution stabilizes and behaves as the process (see [8])

$$Y_t=X_t+\frac{F}{\sqrt{(\omega_0^2-\alpha^2)^2+\alpha^2\gamma^2}}\cos(\alpha t+\beta),$$

where X_t is the stationary solution of the stochastic differential equation (9) and $\tan\beta=\frac{\alpha\gamma}{\omega_0^2-\alpha^2}$.
Thus the zero-crossings of the process Y are the crossings by X of the $2\pi/\alpha$-periodic curve ψ defined by

$$\psi(t)=-\frac{F}{\sqrt{(\omega_0^2-\alpha^2)^2+\alpha^2\gamma^2}}\cos(\alpha t+\beta).$$

Hence, by Proposition 1 we obtain

$$N_t^Y(0) = N_t^X(\psi) = \omega_0 \sum_{q=0}^{\infty} \int_0^t \sum_{l=0}^{q} \kappa_{ql}(s) H_{q-l}\left(\frac{X_s}{\sqrt{m_0}}\right) H_l\left(\frac{\dot{X}_s}{\sqrt{m_2}}\right) ds$$

and therefore

$$\mathbb{E}[N_t^Y(0)] = \omega_0 \int_0^t \varphi\left(\frac{\psi_s}{\sqrt{m_0}}\right) a_0\left(\frac{\dot{\psi}_s}{\sqrt{m_2}}\right) ds.$$

Choosing $t = t(n) = n\frac{2\pi}{\alpha}$ and letting $n \to \infty$ yield via the ergodic theorem

$$\begin{aligned} <\omega>_R = \lim_{t\to\infty} \frac{N_t^Y(0)}{t} &= \omega_0 \frac{\alpha}{2\pi} \int_0^{\frac{2\pi}{\alpha}} \varphi\left(\frac{\psi_s}{\sqrt{m_0}}\right) a_0\left(\frac{\dot{\psi}_s}{\sqrt{m_2}}\right) ds \\ &= \frac{\omega_0}{2\pi} \int_0^{2\pi} \varphi\left(C_1 \cos s\right) a_0\left(C_2 \sin s\right) ds \quad \text{a.s.} \end{aligned}$$

where $C_1 = \frac{-F/\sqrt{m_0}}{\sqrt{(\omega_0^2-\alpha^2)^2+\alpha^2\gamma^2}}$ and $C_2 = \frac{\alpha F/\sqrt{m_2}}{\sqrt{(\omega_0^2-\alpha^2)^2+\alpha^2\gamma^2}}$.

Theorem 1 allows to conclude to the *asymptotical normality of the estimator of* $<\omega>_R$:

$$\lim_{n\to\infty} \sqrt{n}\left(\frac{N_{t(n)}^Y(0)}{n} - \frac{2\pi}{\alpha} <\omega>_R\right) = \mathcal{N}\left(0, \sum_{q=1}^{\infty} \sigma^2(q)\right),$$

where $\sigma^2(q)$ *is computed in Theorem 1, with* $p = 2\pi/\alpha$.

Remark. It would be important to study another interesting problem, that is the asymptotic behavior of the Rice frequency for non-linear oscillators driven by a white noise with a periodic forcing, as for instance the Kramers oscillator defined by

$$\ddot{X}_t + \gamma \dot{X}_t + X_t^3 - X_t = \sigma dW_t + F\cos(\alpha t).$$

It is well known that for such an oscillator, as well as for other Hamiltonian oscillators, the stationary solution is exponentially ergodic hence β-mixing. The asymptotic behavior of the average of zero crossings by X per unit time may be obtained via the ergodic theorem and the Rice formula. But the CLT remains an open problem.

2.2.2 Linear curve and specular points

We are interested in describing the behavior of the specular points in a random curve.
As was pointed out by Longuet-Higgins (see [18]), specular points are the moving images of a light source reflected at different points on a wave-like surface. Let us modelize this surface by a Gaussian field W defined on $\mathbb{R}^+ \times \mathbb{R}$, $\mathbb{R}^+$ for the temporal variable and $\mathbb{R}$ for the spatial one.
The first derivatives with respect to the spatial variable x and the temporal variable t will be denoted by $\partial_x W$ and $\partial_t W$ respectively; the second derivatives will be denoted by $\partial_{xx} W$, $\partial_{tt} W$, $\partial_{tx} W$ and $\partial_{xt} W$.
The spectral representation of W is given by

$$W(t,x) = \int_\Lambda e^{i(\lambda x - \omega t)} \sqrt{f(\lambda)}\, dB(\lambda) = \int e^{i(\lambda x - |\lambda|^{1/2} t)} \sqrt{f(\lambda)}\, dB(\lambda), \tag{10}$$

where $\Lambda = \{(\lambda, \omega) : \omega^2 = \lambda\}$ (the Airy relation) and the real and imaginary parts of the complex-Gaussian process $B = B(\lambda, \lambda \geq 0)$ are real Gaussian processes with $var(Re(B(\lambda))) = var(Im(B(\lambda))) = F(\lambda)/2$ and $B(-\lambda) = \overline{B(\lambda)}$ a.s., having independent increments.
The covariance function writes

$$r(t,x) = 2\int_0^\infty \cos(\lambda x - \lambda^{1/2} t) f(\lambda) d\lambda. \tag{11}$$

First let us consider the process at a fixed time, for instance $t = 0$.
In this case, $W(0, x)$ is a centered stationary Gaussian process with correlation function

$$r(0, x) := r(x) = 2 \int_0^\infty \cos(\lambda x) f(\lambda)\, d\lambda.$$

Let us define a curve in the plan (x, z) by the equation $z = W(0, x)$.
Suppose that the coordinates of a point-source of light and an observer are $(0, h_1)$ and $(0, h_2)$ respectively, situated at heights h_1 and h_2 above the mean surface level. A specular point is characterized by the equations (see [18]):

$$\partial_x W(0, x) = -\kappa x \qquad \text{with} \qquad \kappa = \frac{1}{2}\left(\frac{1}{h_1} + \frac{1}{h_2}\right), \tag{12}$$

which can be interpreted as *a crossing of the curve* $\psi(x) := -\kappa x$ *by the process* $\partial_x W(0, x)$.
Let us assume that r is four times differentiable and that r'' satisfies the Geman condition:

$$\frac{r^{(iv)}(x) - r^{(iv)}(0)}{x} \in L^1\left([0, \delta]\right). \tag{13}$$

Theorem 2 *Under the Geman condition (13), the number* $N_{[0,x]}$ *of specular points in the interval* $[0, x]$ *has the following expansion in* $L^2(\Omega)$*:*

$$N_{[0,x]} = \frac{\gamma}{\eta} \sum_{q=0}^{\infty} \int_0^x F_q\left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma}\right) ds, \tag{14}$$

where $\eta = \sqrt{-r''(0)}$, $\gamma = \sqrt{r^{(iv)}(0)}$ *and*

$$F_q\left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma}\right) := \sum_{l=0}^{q} \frac{H_{q-l}\left(\frac{-\kappa s}{\eta}\right) \varphi\left(\frac{-\kappa s}{\eta}\right)}{(q-l)!}\, a_l\left(\frac{-\kappa}{\gamma}\right) H_{q-l}\left(\frac{\partial_x W(0,s)}{\eta}\right) H_l\left(\frac{\partial_{xx} W(0,s)}{\gamma}\right),$$

with $a_l(.)$ *defined in (4) and* κ *in (12).*
Its expectation is given by

$$\mathbb{E}[N_{[0,x]}] \;=\; \sqrt{\frac{2}{\pi}} \frac{\kappa}{\eta} \left(\int_0^{\frac{\kappa}{\gamma}} e^{-\frac{u^2}{2}}\, du + \frac{\gamma}{\kappa}\, e^{-\frac{\kappa^2}{2\gamma^2}}\right) \int_0^x \varphi\left(\frac{\kappa s}{\eta}\right) ds, \tag{15}$$

and its variance by

$$\begin{aligned} Var(N_{[0,x]}) \;&= \\ \int_0^x \int_0^{x-s} \mathbb{E}&\left[F_q\left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma}\right) F_q\left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma}\right)\right] d\tau ds. \end{aligned}$$

Suppose that the process $\partial_x W(0, x)$ *is* m*-dependent, i.e. that* $-r''(\tau) = 0$ *for* $\tau > m$*, then the asymptotic variance of* $N_{[0,x]}$*, as* $x \to \infty$*, is given by*

$$\begin{aligned} &Var(N_{[0,\infty]}) \\ &= 2\frac{\gamma}{\eta} \sum_{q=1}^{\infty} \int_0^\infty \int_0^m \mathbb{E}\left[F_q\left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma}\right) F_q\left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma}\right)\right] d\tau ds. \end{aligned} \tag{16}$$

Remarks.

i) Note that the expectation (15) of $N_{[0,x]}$ was heuristically obtained by Longuet-Higgins in [18] and [19], who considered a bilateral formula $\mathbb{E}[N_{[-x,x]}]$ and showed that its limit as $x \to \infty$ is given by

$$\lim_{\frac{\gamma}{\kappa}\to\infty} \frac{\kappa}{\gamma}\,\mathbb{E}[N_{[0,+\infty]}] = \sqrt{\frac{1}{2\pi}} \quad ;$$

it can be interpreted by saying that the number of specular points increases proportionally to the distance between the observer and the sea surface, since when $h_1 = h_2$, $\frac{1}{\kappa}$ represents this latter distance.
It is also interesting to notice that the number of specular points is a finite random variable over all the line, on the contrary to the behavior of the crossings of a fixed level or to the one of the number of maxima.

ii) We consider the hypothesis of m-dependence to simplify the computations and with the aim of using in the next section the method settled in [14].

iii) The computations may be done in dimension 2, explicitly in what concerns the expectation, but we would face some difficulty for the second moment since the determinant appearing in the integrand of it would prevent from having a Hermite polynomial expansion.

3 Speed of specular points on the sea surface

Let $\tilde{W}(t,x,y)$ represent the waves modeling the sea surface. We shall look at a wave in one direction, that is for fixed coordinate y, for instance when $y = 0$. Thus we consider from now on $W(t,x) = \tilde{W}(t,x,0)$.
Our main goal is the study of the number of specular points having a given velocity.
To represent such a number into the Itô-Wiener chaos will require a generalization of Rice formula to the bidimensional case, obtained by using the co-area formula (see [6]) that we recall here.
Let W be a three times continuously differentiable bivariate function ($W \in \mathcal{C}^3$), and let consider the level curve $\mathcal{C}(u)$ defined by

$$\mathcal{C}(u) = \{(x_1,x_2) \in Q(t,M) : W(x_1,x_2) = u\}, \quad \text{where} \quad Q(t,M) = [0,t]\times[0,M].$$

If v denotes a vector field, $\mathbf{n}$ denotes the vector normal to the level curve $\mathcal{C}(u)$ and $d\sigma$ the length measure of $\mathcal{C}(u)$, since W is $\mathcal{C}^3$, then the Green formula provides for some suitable function g,

$$\int_{-\infty}^{\infty} g(u)\int_{\mathcal{C}(u)} < v, \mathbf{n} > d\sigma\; du = \int_{Q(t,M)} g(W) < v, \nabla W > dx_1\, dx_2,$$

from which can be immediately deduced:

i) if $v = \frac{\nabla W}{||\nabla W||}$, then $\int_{-\infty}^{\infty} g(u)\mathcal{L}_Q(\mathcal{C}(u))du = \int_Q g(W)||\nabla W||dx_1\, dx_2$,
where $\mathcal{L}_Q(\mathcal{C}(u))$ denotes the length of the curve $\mathcal{C}(u)$;

ii) more generally, if $v = \zeta(\alpha)\frac{\nabla W}{||\nabla W||}$, where ζ denotes a continuous real function defined on $[0,2\pi]$ and α is defined by $\nabla W := ||\nabla W||\;^t(\cos\alpha, \sin\alpha)$, then

$$\int_{-\infty}^{\infty} g(u)\int_{\mathcal{C}(u)} \zeta(\alpha(s))d\sigma(s)du = \int_Q g(W)\zeta(\alpha)||\nabla W||dx_1\, dx_2.$$

Note that this formula still holds when considering ζ as the indicator function of a measurable set, by using the monotone convergence theorem (see for instance [7] and [2]).

Let us go back to our study and keep the same notation as in the previous section.
Suppose W and r satisfy (10) and (11) respectively, with mixed spectral moment m_{pq} defined by

$$m_{pq} = 2\int_{\{(\lambda,\omega):\omega^2=\lambda\}} \lambda^p \omega^q f(\lambda)d\lambda = 2\int_0^\infty \lambda^{p+q/2} f(\lambda)d\lambda. \tag{17}$$

Note that $(\partial_x W(t,x), \partial_{xx}W(t,x), \partial_{tx}W(t,x))^t$ is Gaussian, with covariance matrix $\Delta = (\Delta_{ij})_{1\le i,j\le 3}$, such that

$$\begin{array}{lcr} \Delta_{11} = \mathbb{E}[\partial_x W^2] = m_{20} & \Delta_{22} = \mathbb{E}[\partial_{xx} W^2] = m_{40} & \Delta_{33} = \mathbb{E}[\partial_{xt} W^2] = m_{22} \\ \Delta_{12} = \mathbb{E}[\partial_x W \partial_{xx} W] = 0 = \Delta_{13} = \mathbb{E}[\partial_x W \partial_{xt} W] & \text{and} & \Delta_{23} = \mathbb{E}[\partial_{xx} W \partial_{xt} W] = m_{31}. \end{array}$$

Moreover, for fixed (t,x), $\partial_x W(t,x)$ is independent of the vector $(\partial_{xx}W(t,x), \partial_{xt}W(t,x))$.
As Longuet-Higgins (see [19]), we consider the following simplified condition to have a specular point:

$$\partial_x W(t,x) = u.$$

A straight consequence of the implicit function theorem is:

$$\partial_{xx} W dx + \partial_{xt} W dt = 0, \qquad \text{i.e. } \frac{dx}{dt} = -\frac{\partial_{xt} W}{\partial_{xx} W}.$$

We are interested in the number of specular points with bounded speed: let $-\,v_2$ denote the lower bound and $-\,v_1$ the upper one. We define

$$\tilde{N}_{sp}(s,u,v_1,v_2) := \#\left\{ z \in [0,M] : \partial_x W(s,z) = u;\ v_1 \le \frac{\partial_{xt} W}{\partial_{xx} W} \le v_2 \right\}, \qquad \text{for } 0 \le s \le t,$$

and also

$$N_{sp}(u,v_1,v_2,t) := \frac{1}{t}\int_0^t \tilde{N}_{sp}(s,u,v_1,v_2)ds, \tag{18}$$

since our interest is in that number per unit time.

Lemma 1 *The expected number μ of specular points in $Q(t,M)$ with speed belonging to $[-v_2,-v_1]$ is given by*

$$\mu := \mathbb{E}\left[N_{sp}(u,v_1,v_2,t)\right] = M\frac{e^{-\frac{u^2}{2m_{20}}}\lambda_1\lambda_2}{4\pi\sqrt{m_{20}}}\int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta-\beta)| \frac{1}{(\lambda_1 \sin^2\theta + \lambda_2\cos^2\theta)^{3/2}}\, d\theta,$$

where m_{pq} is defined in (17), λ_1, λ_2 are the eigenvalues of the covariance matrix of $(\partial_{xx}W(0,0), \partial_{xt}W(0,0))$, and β is the rotation angle which turns diagonal this covariance matrix.

Remark. If the random field $W(t,x)$ is isotropic the vector $(\partial_{xx}W(0,0), \partial_{xt}W(0,0))$ has a diagonal covariance matrix, thus $\beta = 0$ and $\lambda_1 = \lambda_2$ and we obtain when supposing e.g. that $\lambda_1 = \lambda_2 = 1$,

$$\mu = M\frac{e^{-\frac{u^2}{2m_{20}}}}{4\pi m_{20}}\left[\frac{v_2}{\sqrt{1+v_2^2}} - \frac{v_1}{\sqrt{1+v_1^2}}\right], \quad \text{if} \quad 0 \le \arctan v_1 \le \arctan v_2 \le \frac{\pi}{2}.$$

Proof of Lemma 1.
Let g be a continuous and bounded function. We have

$$\begin{aligned} \int_{-\infty}^{\infty} g(u) N_{sp}(u,v_1,v_2,t)du &= \frac{1}{t}\int_0^t \int_{-\infty}^{\infty} g(u)\tilde{N}_{sp}(s,u,v_1,v_2)duds \\ &= \frac{1}{t}\int_0^t\int_0^M g(\partial_x W(s,z)) \mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(s,z)}{\partial_{xx}W(s,z)}\right) |\partial_{xx}W(s,z)|\, dzds \\ &= \frac{1}{t}\int_{-\infty}^{\infty} g(u)\int_{\mathcal{C}(u)} \zeta(\alpha(s))d\sigma(s)du, \quad \text{with} \quad \zeta(\alpha(s)) := \mathbb{1}_{[v_1,v_2]}(\tan\alpha(s))|\cos\alpha(s)|, \end{aligned}$$

by using the Banach formula in the first equality and the co-area formula in the last one.
Note that this type of integrals has been first considered by Cabaña (see [7]).
Therefore, by independence and stationarity, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} g(u)\mathbb{E}\left[N_{sp}(u,v_1,v_2,t)\right]du &= \frac{1}{t}\int_{-\infty}^{\infty} g(u)\mathbb{E}\left[\int_{\mathcal{C}(u)} \zeta(\alpha(s))d\sigma(s)\right]du \\ &= M\int_{-\infty}^{\infty} g(u)\frac{e^{-\frac{u^2}{2m_{20}}}}{\sqrt{2\pi m_{20}}}du\mathbb{E}\left[\mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(0,0)}{\partial_{xx}W(0,0)}\right)|\partial_{xx}W(0,0)|\right],\end{aligned}$$

and by duality it comes that, for almost all u,

$$\mathbb{E}\left[N_{sp}(u,v_1,v_2,t)\right] = M\frac{e^{-\frac{u^2}{2m_{20}}}}{\sqrt{2\pi m_{20}}}\mathbb{E}\left[\mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(0,0)}{\partial_{xx}W(0,0)}\right)|\partial_{xx}W(0,0)|\right]. \tag{19}$$

It can be proved that (19) holds for all u (see [6] and [7]).

Note that at fixed (s,z), in particular at $(0,0)$, the covariance matrix of the random vector $(\partial_{xx}W(s,z),\partial_{xt}W(s,z))$ can be diagonalized with a rotation matrix of angle β, independent of the point (s,z) because of the stationarity of the process $(\partial_{xx}W(.,.),\partial_{xt}W(.,.))$, thus we can write, for fixed (s,z),

$$\begin{pmatrix}\partial_{xx}W(s,z)\\ \partial_{xt}W(s,z)\end{pmatrix} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\eta_1(s,z)\\ \eta_2(s,z)\end{pmatrix}, \tag{20}$$

with $(\eta_1(s,z),\eta_2(s,z))$ normally distributed $\mathcal{N}\left(0,\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\right)$.
Computing the expectation of the RHS of (19) gives

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(0,0)}{\partial_{xx}W(0,0)}\right)|\partial_{xx}W(0,0)|\right] &= \frac{1}{2\pi\sqrt{\lambda_1\lambda_2}}\int_0^{\infty}\int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta-\beta)|\, r^2 e^{-\frac{r^2}{2}\left(\frac{\cos^2\theta}{\lambda_1}+\frac{\sin^2\theta}{\lambda_2}\right)}d\theta dr \\ &= \frac{\lambda_1\lambda_2}{\sqrt{8\pi}}\int_{\beta+\arctan v_1}^{\beta+\arctan v_2} |\cos(\theta-\beta)|\,\frac{1}{(\lambda_1\sin^2\theta+\lambda_2\cos^2\theta)^{3/2}}d\theta,\end{aligned}$$

hence the result for μ given in the lemma. □

Now we are looking at the Hermite expansion of $N_{sp}(u,v_1,v_2,t)$ (defined in (18)) and its asymptotic behavior as $t\to\infty$.

Theorem 3 *If $W(s,z)\in\mathcal{C}^3$, then the Hermite expansion of the functional $N_{sp}(u,v_1,v_2,t)$ is given by*

$$\begin{aligned}N_{sp}(u,v_1,v_2,t) &= \sum_{l=0}^{\infty}\sum_{0\le n+m\le l}\frac{H_{l-(n+m)}\left(\frac{u}{\sqrt{m_{20}}}\right)\varphi\left(\frac{u}{\sqrt{m_{20}}}\right)g_{n,m}}{\sqrt{m_{20}}\,[l-(n+m)]!} \\ &\qquad\times\frac{1}{t}\int_0^t\int_0^M H_{l-(n+m)}\left(\frac{\partial_x W(s,z)}{\sqrt{m_{20}}}\right)H_m\left(\frac{\eta_1(s,z)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(s,z)}{\sqrt{\lambda_2}}\right)dzds,\end{aligned} \tag{21}$$

where $(\eta_1(s,z),\eta_2(s,z))$ is normally distributed $\mathcal{N}\left(0,\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\right)$ with λ_1,λ_2 defined in Lemma 1, and where

$$g_{n,m} = \frac{1}{2\pi n!m!}\int_0^{\infty}\int_{\arctan b_1}^{\arctan b_2}\left|\sqrt{\lambda_1}\cos\beta\cos\theta-\sqrt{\lambda_2}\sin\beta\sin\theta\right| H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho^2 d\theta d\rho, \tag{22}$$

with

$$b_i := \sqrt{\frac{\lambda_1}{\lambda_2}}\left(\frac{v_i-\tan\beta}{1+v_i\tan\beta}\right), \quad i=1,2. \tag{23}$$

Moreover, assuming that $W(t,s)$ is m-dependent in the time variable t, we have

$$\sqrt{t}\,(N_{sp}(u,v_1,v_2,t)\ -\ \mu)\ \xrightarrow{d}\ \mathcal{N}(0,\sigma^2(M)),\quad ast\to\infty,$$

with μ defined in Lemma 1 and $\sigma^2(M)$ computed in (27) below.

Remark. If $W(t,x)$ is isotropic, the coefficients (22) simplify as

$$\begin{aligned} g_{n,m} &= \frac{1}{2\pi n!m!}\int_0^\infty\int_{\arctan v_1}^{\arctan v_2} |\cos\theta|\, H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho^2 d\theta d\rho \\ &= \frac{1}{2\pi n!m!}\int_0^\infty\int_{\frac{v_1}{\sqrt{1-v_1^2}}}^{\frac{v_2}{\sqrt{1-v_2^2}}} H_n(\rho\sqrt{1-x^2})H_m(\rho x)e^{-\frac{\rho^2}{2}}\rho^2 dxd\rho. \end{aligned}$$

This last integral could be computed explicitly as done in [14] (see p. 663, proof of Lemma 5).

Proof. The proof of Theorem 3 relies mainly on the application of the co-area formula and follow the method developed in [14]. Suppose w.l.o.g. $\mathbb{E}\left(\partial_x^2 W(t,z)\right) = m_{20} = 1$.

i) Hermite expansion for $N_{sp}(u,v_1,v_2)$.
It can be shown that

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\zeta(\alpha(s))d\sigma(s)\right]dy \underset{h\to 0}{\to} \int_{\mathcal{C}(u)}\zeta(\alpha(s))d\sigma(s) \ =\ tN_{sp}(u,v_1,v_2),\qquad \text{in } L^2(\Omega),$$

which will be useful to obtain the Hermite expansion of $N_{sp}(u,v_1,v_2)$.
Using the co-area formula yields

$$\begin{aligned} &\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\zeta(\alpha(s))d\sigma(s)\right]dy \\ &\qquad = \frac{1}{h}\int_0^t\int_0^M \varphi\left(\frac{u-\partial_x W(s,z)}{h}\right)\mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(s,z)}{\partial_{xx}W(s,z)}\right)|\partial_{xx}W(s,z)|\,dzds. \end{aligned}$$

Using (20) allows to write

$$\begin{aligned} \mathbb{1}_{[v_1,v_2]}\left(\frac{\partial_{xt}W(s,z)}{\partial_{xx}W(s,z)}\right)|\partial_{xx}W(s,z)| &= \mathbb{1}_{[v_1,v_2]}(\tan(\beta+\eta(s,z)))\ |\eta_1(s,z)\cos\beta-\eta_2(s,z)\sin\beta| \\ &= \mathbb{1}_{\left[\frac{v_1-\tan\beta}{1+v_1\tan\beta},\frac{v_2-\tan\beta}{1+v_2\tan\beta}\right]}\left(\frac{\eta_2(s,z)}{\eta_1(s,z)}\right)\ |\eta_1(s,z)\cos\beta-\eta_2(s,z)\sin\beta| \\ &= \mathbb{1}_{[b_1,b_2]}\left(\frac{Z_1(s,z)}{Z_2(s,z)}\right)\left|\sqrt{\lambda_1}\ Z_1(s,z)\cos\beta-\sqrt{\lambda_2}\ Z_2(s,z)\sin\beta\right| \\ &:= G(Z_1,Z_2)(s,z) \end{aligned}$$

where $\eta(s,z) := \arctan\left(\dfrac{\eta_2(s,z)}{\eta_1(s,z)}\right)$, $Z_i(s,z) := \left(\dfrac{\eta_i(s,z)}{\sqrt{\lambda_i}}\right)$, $(i=1,2)$, are normally distributed $\mathcal{N}(0,Id)$ for fixed (s,z), and where b_i, $i=1,2$, are defined in (23).
The Hermite coefficients $g_{n,m}$ of the functional $G(Z_1,Z_2)$ can be computed as

$$\begin{aligned} g_{n,m} &= \frac{1}{n!m!}\int_{\mathbb{R}^2} G(z_1,z_2)H_n(z_1)H_m(z_2)\varphi(z_1)\varphi(z_2)dz_1dz_2 \\ &= \frac{1}{2\pi n!m!}\int_0^\infty\int_0^{2\pi} G(\rho\cos\theta,\rho\sin\theta)H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho d\theta d\rho \\ &= \frac{1}{2\pi n!m!}\int_0^\infty\int_{\arctan b_1}^{\arctan b_2}\left|\sqrt{\lambda_1}\cos\beta\cos\theta-\sqrt{\lambda_2}\sin\beta\sin\theta\right| H_n(\rho\cos\theta)H_m(\rho\sin\theta)e^{-\frac{\rho^2}{2}}\rho^2 d\theta d\rho, \end{aligned}$$

i.e. (22). So we obtain

$$\frac{1}{h}\int_{-\infty}^{\infty}\varphi\left(\frac{u-v}{h}\right)\left[\int_{\mathcal{C}(v)}\zeta(\alpha(s))d\sigma(s)\right]du$$

$$=\sum_{l=0}^{\infty}\sum_{0\le n+m\le l}\frac{c_{l-(n+m)}(u,h)g_{n,m}}{[l-(n+m)]!}\int_0^t\int_0^M H_{l-(n+m)}(\partial_x W(s,z))H_m\left(Z_1(s,z)\right)H_n\left(Z_2(s,z)\right)dzds,$$

where the Hermite coefficients $c_k(y,h)$ of the function $\frac{1}{h}\varphi\left(\frac{\cdot\ -\ y}{h}\right)$ are given by

$$c_k(y,h)=\frac{1}{k!}\int_{-\infty}^{\infty}\varphi(v)H_k(y-hv)\varphi(y-hv)dv \underset{h\to 0}{\to} \frac{H_k(y)\varphi(y)}{k!}. \tag{24}$$

Now we can deduce the Hermite expansion of $N_{sp}(u,v_1,v_2,t)$ as in [14], namely

$$\frac{1}{h\ t}\int_{-\infty}^{\infty}\varphi\left(\frac{u-y}{h}\right)\left[\int_{\mathcal{C}(y)}\zeta(\alpha(s))d\sigma(s)\right]dy \quad \underset{h\to 0}{\to} \quad N_{sp}(u,v_1,v_2), \quad \text{in } L^2(\Omega),$$

where $N_{sp}(u,v_1,v_2,t)=$

$$\sum_{l=0}^{\infty}\sum_{0\le n+m\le l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{[l-(n+m)]!}\frac{1}{t}\int_0^t\int_0^M H_{l-(n+m)}(\partial_x W(s,z))H_m\left(\frac{\eta_1(s,z)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(s,z)}{\sqrt{\lambda_2}}\right)dzds.$$

ii) CLT for $N_{sp}(u,v_1,v_2,t)$.
Suppose that the field $W(s,x)$ is m-dependent in time, that is its correlation function satisfies $r(s,x)=0$ whenever $s>m$, for all x.
According to our general method (see [14]), the proof consists mainly in studying the asymptotic behavior of an L^2-approximation of $N_{sp}(u,v_1,v_2,t)$ defined by the finite expansion

$$\begin{aligned} N_{sp}^Q(u,v_1,v_2,t) \quad &:= \quad \sum_{l=0}^{Q}\sum_{0\le n+m\le l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{(l-(n+m))!} \\ &\times\frac{1}{t}\int_0^t\int_0^M H_{l-(n+m)}(\partial_x W(s,z))H_m\left(\frac{\eta_1(s,z)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(s,z)}{\sqrt{\lambda_2}}\right)dzds. \end{aligned} \tag{25}$$

$N_{sp}^Q(u,v_1,v_2,t)$ can be written in terms of the occupation functional

$$S_t(M)=\int_0^t\int_0^M F_Q(\mathbf{W}(s,z))dzds=\int_0^t\int_0^M F_Q\circ\mathbf{\Sigma}^{1/2}(\mathbf{Y}(s,z))dzds,$$

where F_Q is a polynomial function of order Q, $\mathbf{W}$ is the Gaussian random field defined by $\mathbf{W}(t,x)=(\partial_x W(t,x),\partial_{xx}W(t,x),\partial_{tx}W(t,x))^t$ and where

$$\mathbf{Y}(t,x):=\mathbf{\Sigma}^{-1/2}\mathbf{W}(t,x),$$

with $\mathbf{R}(t,x):=\mathbb{E}\left[\mathbf{W}(t,x)(\mathbf{W}(0,0))^t\right]$ and $\mathbf{\Sigma}:=\mathbf{R}(0,0)=\mathbb{E}\left[\mathbf{W}(0,0)(\mathbf{W}(0,0))^t\right]$.
Note that $F_Q\circ\mathbf{\Sigma}^{1/2}$ is also a polynomial function and that the field $\mathbf{Y}(t,x):=(Y_i(t,x),i=1,2,3)$ thus defined is m-dependent in time.
Given that $F_Q\circ\mathbf{\Sigma}^{1/2}\in L^2(\phi(x_1)\phi(x_2)\phi(x_3)dx_1dx_2dx_3)$, we have

$$S_t(M) \quad = \quad \sum_{|\mathbf{k}|\le Q}c_{\mathbf{k}}\int_0^t\int_0^M H_{k_1}(Y_1(s,z))H_{k_2}(Y_2(s,z))H_{k_3}(Y_3(s,z))dzds,$$

where $\mathbf{k}=(k_1,k_2,k_3)$, $|\mathbf{k}|=k_1+k_2+k_3$ and $c_{\mathbf{k}}$ are the Hermite coefficients of $F_Q\circ\mathbf{\Sigma}^{1/2}$.
Let us check that

$$\frac{S_t(M)}{t}\underset{t\to\infty}{\longrightarrow} c_{\mathbf{0}} \qquad \text{in probability.} \tag{26}$$

Defining

$$I(|\mathbf{k}|, k_2, k_3, s, z) := H_{|\mathbf{k}|-(k_2+k_3)}(Y_1(s,z))H_{k_2}(Y_2(s,z))H_{k_3}(Y_3(s,z)),$$

we can write

$$S_t(M) = \sum_{|\mathbf{k}|=0}^{Q} \sum_{0\le k_2+k_3\le|\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3),\,k_2,\,k_3)} \int_0^t \int_0^M I(|\mathbf{k}|, k_2, k_3, s, z)dzds.$$

The random variables $(Y_i(0,0), 1 \le i \le 3)$ being independent standard Gaussian, Parseval's equality gives

$$\mathbb{E}\left([F_Q \circ \mathbf{\Sigma}^{1/2}(\mathbf{Y}(0,0))]^2\right) = \sum_{|\mathbf{k}|\le Q} c_{\mathbf{k}}^2 k_1!k_2!k_3! < \infty.$$

Therefore, using the Diagram formula and the Dominated Convergence Theorem, we obtain for $t > m$

$$\begin{aligned}\sigma^2(M) := t\mathbb{E}\left[\left(\frac{S_t(M)}{t} - c_{\mathbf{0}}\right)^2\right] &= 2\sum_{|\mathbf{k}|=1}^{Q} \sum_{0\le k_2+k_3\le|\mathbf{k}|} \sum_{0\le l_2+l_3\le|\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3),\,k_2,\,k_3)} c_{(|\mathbf{k}|-(l_2+l_3),\,l_2,\,l_3)} \\ &\quad \int_0^m \int_0^M (M-z)\mathbb{E}[I(|\mathbf{k}|, k_2, k_3, s, z)I(|\mathbf{k}|, l_2, l_3, 0, 0)]dzds. \qquad (27)\end{aligned}$$

Hence the result (26).
It can be proved that the almost sure convergence holds too.

Let us study the weak limit of the sequence $\sqrt{t}\left(\frac{S_t(M)}{t} - c_{\mathbf{0}}\right)$.
Defining $\tilde{S}(s,M) = \sum_{|\mathbf{k}|=1}^{Q} \sum_{0\le k_2+k_3\le|\mathbf{k}|} c_{(|\mathbf{k}|-(k_2+k_3),\,k_2,\,k_3)} \int_0^M I(|\mathbf{k}|, k_2, k_3, s, z)dz$, we have

$$\begin{aligned}\sqrt{t}\left(\frac{S_t(M)}{t} - c_{\mathbf{0}}\right) &= \frac{1}{\sqrt{t}}\sum_{k=1}^{[t]} \int_{(k-1)}^{k} \tilde{S}(s,M)ds + \frac{1}{\sqrt{t}}\int_{[t]}^{t} \tilde{S}(s,M)ds \\ &= \frac{1}{\sqrt{t}}\sum_{k=1}^{[t]} \theta_{k-1} \circ \int_0^1 \tilde{S}(s,M)ds + \frac{1}{\sqrt{t}}\int_{[t]}^{t} \tilde{S}(s,M)ds, \qquad (28)\end{aligned}$$

when introducing the time shift operator θ_h.
Thus (28) appears as a sum of m-dependent random variables having second moment, to which the Hoeffding-Robbins Theorem (see [10]) can be applied, providing

$$\sqrt{t}\left(\frac{S_t(M)}{t} - c_{\mathbf{0}}\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(M)) \quad \text{as } t \to \infty,$$

where $c_0 = \mu$ and $\sigma^2(M)$ is defined in (27),

$$\text{i.e.} \quad \sqrt{t}\left(N_{sp}^Q(u, v_1, v_2, t) - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(M)) \quad \text{as } t \to \infty. \qquad (29)$$

To conclude the proof we verify that

$$\limsup_{Q\to\infty} \limsup_{t\to\infty} \mathbb{E}\left([\sqrt{t}\left(N_{sp}(u, v_1, v_2, t) - N_{sp}^Q(u, v_1, v_2, t)\right)]^2\right) = 0. \qquad (30)$$

Indeed, we can write

$$\sqrt{t}\left(N_{sp}(u, v_1, v_2, t) - N_{sp}^Q(u, v_1, v_2, t)\right) = \frac{1}{\sqrt{t}}\sum_{k=1}^{[t]} \theta_{k-1} \circ Y_Q(M) + \frac{1}{\sqrt{t}} I_2(t), \qquad (31)$$

where

$$Y_Q(M)=\sum_{l=Q+1}^{\infty}\sum_{0\le n+m\le l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{(l-(n+m))!}\int_0^1\int_0^M H_{l-(n+m)}(\partial_x W(s,z))H_m\left(\frac{\eta_1(s,z)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(s,z)}{\sqrt{\lambda_2}}\right)dzds$$

and

$$I_2(t):=\sum_{l=Q+1}^{\infty}\sum_{0\le n+m\le l}\frac{H_{l-(n+m)}(u)\varphi(u)g_{n,m}}{(l-(n+m))!}\int_{[t]}^t\int_0^M H_{l-(n+m)}(\partial_x W(s,z))H_m\left(\frac{\eta_1(s,z)}{\sqrt{\lambda_1}}\right)H_n\left(\frac{\eta_2(s,z)}{\sqrt{\lambda_2}}\right)dzds,$$

The first term in (31) is a sum of m-dependent random variables whose asymptotic variance tends to zero as Q tends to infinity; the second term in (31) tends also to zero as $t\to\infty$ uniformly in Q since we have

$$Var(I_2(t))\ \le\ \mathbb{E}\left[(N_{sp}(u,v_1,v_2,0))^2\right]\ =\ O(1).$$

Hence (30) is satisfied. □

Remark. Note that the proportion $\frac{N_{sp}(u,v_1,v_2,t)}{N_{[0,M]}}$ of the number of specular points with a given velocity (e.g. between $-v_2$ and v_1) among the number of specular points can be shown to converge to a Gaussian r.v. as it was done in [13] for the number of maxima.

4 Remark

We are pursuing our study on characteristics of sea waves, in particular on twinkles. Longuet-Higgins has shown in [18] that the specular points evolve until a certain time up to the vanishing of the curvature. The number of images seen by the observer is not constant; specular points may appear or disappear, the images move. The creation or annihilation of specular points may be called a twinkle. At such a twinkle, the water surface is not only oriented to reflect light into the eye, but is curved so as to focuse it there, which can be mathematically translated as

$$\begin{cases} Y_1(t,x)=\partial_x f(t,x)=0 & : \text{ to have a specular point}\\ Y_2(t,x)=\partial_{xx} f(t,x)=0 & : \text{ to have a singularity in the curvature,}\end{cases}$$

when introducing the function $f(t,x)=W(t,x)+\dfrac{1}{2}\kappa x^2$, $(t,x)\in Q(t,M)$, and where $W(t,x)$ is the process whose spectral representation is given in (10).

Using a multidimensional Rice formula to count the number of roots of a nonlinear system of equations having the same number of equations and variables (see [6], p.80, or [3]) and introducing a regression model allowed us to compute explicitly the mean number of twinkles at fixed time t and when $M\to\infty$, as $\mathbb{E}[N_{TW}]=\dfrac{t}{\sqrt{2\pi^3}}\dfrac{\sqrt{m_{60}\,(m_{40}m_{22}-m_{31}^2)}}{\kappa\ m_{40}}\,e^{-\frac{\kappa^2}{2m_{40}}}\left[e^{-\frac{a^2}{2}}+a\displaystyle\int_0^a e^{-\frac{v^2}{2}}dv\right]$, where m_{pq} is defined in (17) and $a:=\dfrac{m_{31}}{\sqrt{m_{40}\,(m_{40}m_{22}-m_{31}^2)}}$. Note that it corresponds to Longuet-Higgins's heuristic formula (see [18], p.853), except to a factor 2 that would correspond in our case to work on $[0,t]\times[-M,M]$ rather than on $Q(t,M)$, with $M\to\infty$.

Computing the variance and looking for a CLT is still an ongoing research.

Appendix

A1 - Proof of Proposition 1

As already mentioned, we follow the main approach we proposed in [12] to obtain the result for a fixed level. Technical tools developed recently in [15] allow, now, to generalize this result to differentiable curves.

The proof can be sketched in four main steps, outlined below for the paper to be self-contained.

• *Smooth approximation of X_t.*
Consider a twice differentiable even density function ϕ with support in $[-1,1]$ and define the continuous twice differentiable smoothed process as

$$X_t^\varepsilon = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right) X_u\, du.$$

Let $N_{\varepsilon,t}^{\psi}$ denote the number of ψ_t^ε - crossings by X_t^ε:

$$N_{\varepsilon,t}^{\psi} = card\{s \le t : X_s^\varepsilon = \psi_s^\varepsilon\}, \quad \text{where} \quad \psi_t^\varepsilon = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \phi\left(\frac{t-u}{\varepsilon}\right)\psi_u\, du.$$

Let B be a complex Brownian motion such that $\mathbb{E}\left[B(d\lambda)\overline{B(d\lambda')}\right] = F(d\lambda)\mathbb{1}_{(\lambda=\lambda')}$, F being the spectral measure.
We can write

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} B(d\lambda) \quad \text{and} \quad X_t^\varepsilon = \int_{-\infty}^{\infty} e^{it\lambda}\hat{\phi}(\varepsilon\lambda) B(d\lambda),$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.
The correlation function of the process X_t^ε given by $r_\varepsilon(\tau) = \displaystyle\int_{-\infty}^{\infty} e^{i\tau\lambda}|\hat{\phi}(\varepsilon\lambda)|^2 F(d\lambda)$ satisfies

$$r_\varepsilon(0) \underset{\varepsilon\to 0}{\to} r(0) = \int_{-\infty}^{\infty} F(d\lambda) = 1,$$

and

$$r_\varepsilon(\tau) \;=\; r_\varepsilon(0) + \frac{r_\varepsilon''(0)}{2}\,\tau^2 + \theta_{X^\varepsilon}(\tau),$$

with θ_{X^ε} satisfying the same conditions as θ in (1).
We will work with the normalized process $\tilde{X}_t^\varepsilon = \dfrac{X_t^\varepsilon}{\sigma_\varepsilon}$, where $\sigma_\varepsilon := \sqrt{r_\varepsilon(0)}$, since the number of $\tilde{\psi}_t^\varepsilon$ - crossings by $\tilde{X}_t^\varepsilon$ is the number of ψ_t^ε - crossings by X_t^ε:

$$card\{s \le t : \tilde{X}_s^\varepsilon = \tilde{\psi}_s^\varepsilon\} = N_{\varepsilon,t}^{\psi}, \quad \text{where} \quad \tilde{\psi}_t^\varepsilon = \frac{\psi_t^\varepsilon}{\sigma_\varepsilon}.$$

Note that $\tilde{X}_t^\varepsilon$ is of variance 1 and correlation function ρ_ε such that

$$\rho_\varepsilon(\tau) \;=\; 1 + \frac{\rho_\varepsilon''(0)}{2}\,\tau^2 + \theta_\varepsilon(\tau),$$

where $\rho_\varepsilon''(\tau) = \dfrac{r_\varepsilon''(\tau)}{r_\varepsilon(0)}$ and $\theta_\varepsilon(\tau) = \dfrac{\theta_{X^\varepsilon}(\tau)}{r_\varepsilon(0)}$.
The smoothed process X_t^ε has the fourth derivative of its correlation function r_ε finite in 0, and so does $\tilde{X}_t^\varepsilon$, since these processes are twice differentiable:

$$r_\varepsilon^{(iv)}(0) < \infty \qquad \text{and} \qquad \rho_\varepsilon^{(iv)}(0) < \infty,$$

• *Hermite expansion for $N_{\varepsilon,t}^{\psi}$.*
By using the same approach as in [12], the Hermite expansion for $N_{\varepsilon,t}^{\psi}$ is obtained as $N_{\varepsilon,t}^{\psi} = \lim\limits_{h\to 0} N_{t,h}^{X^\varepsilon}(\psi)$, where

$$N_{t,h}^{X^\varepsilon}(\psi) := \frac{1}{h}\int_0^t \varphi\left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h}\right)\frac{|\dot{X}_s^\varepsilon - \dot{\psi}_s^\varepsilon|}{\sigma_\varepsilon} ds = \frac{\eta_\varepsilon}{h\sigma_\varepsilon}\int_0^t \varphi\left(\frac{\tilde{X}_s^\varepsilon - \tilde{\psi}_s^\varepsilon}{h}\right)\left|\frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon} - \frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon}\right| ds,$$

with φ the standard Gaussian density, $\eta_\varepsilon := \sqrt{-r_\varepsilon''(0)}$ and $h > 0$.
We can write in $L^2(\Omega)$,

$$N_{t,h}^{X^\varepsilon}(\psi) \;=\; \frac{\eta_\varepsilon}{\sigma_\varepsilon}\sum_{q=0}^{\infty}\int_0^t \left[\sum_{l=0}^{q} c_{q-l}(\tilde{\psi}_s^\varepsilon, h)\, a_l\left(\frac{\dot{\psi}_s^\varepsilon}{\eta_\varepsilon}\right) H_{q-l}(\tilde{X}_s^\varepsilon) H_l\left(\frac{\dot{X}_s^\varepsilon}{\eta_\varepsilon}\right)\right] ds, \tag{32}$$

where the Hermite coefficients $a_l(m)$ are defined in (4) and $c_k(y,h)$ in (24).
The expansion of $N^{\psi}_{\varepsilon,\,t}$ follows by taking $h\to 0$ in (32):

$$N^{\psi}_{\varepsilon,\,t}=\frac{\eta_\varepsilon}{\sigma_\varepsilon}\sum_{q=0}^{\infty}\int_0^t\sum_{l=0}^{q}\frac{H_{q-l}\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)\varphi\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)}{(q-l)!}\,a_l\left(\frac{\dot{\psi}^\varepsilon_s}{\eta_\varepsilon}\right)H_{q-l}\left(\frac{X^\varepsilon_s}{\sigma_\varepsilon}\right)H_l\left(\frac{\dot{X}^\varepsilon_s}{\eta_\varepsilon}\right)ds. \tag{33}$$

• *Convergence in $L^2(\Omega)$ of $N^{\psi}_{\varepsilon,\,t}$ to $N^X_t(\psi)$ as $\varepsilon\to 0$.*
To get this L^2- convergence, we will show that

$$\lim_{\varepsilon\to 0}\mathbb{E}[N^{\psi}_{\varepsilon,\,t}]^2=\mathbb{E}[N^X_t(\psi)]^2 \tag{34}$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}[N^X_t(\psi)N^{\psi}_{\varepsilon,\,t}]=\mathbb{E}[N^X_t(\psi)]^2. \tag{35}$$

Proving (34) only requires some work since (35) is easily obtained as in [12] (p.245).
It comes back to prove the convergence of the second factorial moment since we have, by using (33) and the uniform convergence,

$$\mathbb{E}[N^{\psi}_{\varepsilon,\,t}]=\frac{\eta_\varepsilon}{\sigma_\varepsilon}\int_0^t\varphi\left(\frac{\psi^\varepsilon_s}{\sigma_\varepsilon}\right)a_0\left(\frac{\dot{\psi}^\varepsilon_s}{\eta_\varepsilon}\right)ds\ \underset{\varepsilon\to 0}{\to}\ \eta\int_0^t\varphi(\psi_s)a_0\left(\frac{\dot{\psi}_s}{\eta}\right)ds=\mathbb{E}[N^X_t(\psi)].$$

The second factorial moment $M_2^{\varepsilon,\,\psi}$ of the number of $\tilde{\psi}^\varepsilon$-crossings by $\tilde{X}^\varepsilon$ can be deduced from the one of the number of zero-crossings by Y (see [9], p.209), namely

$$M_2^{\varepsilon,\,\psi}=\int_0^t\int_0^t dt_1dt_2\int_{I\!R^2}\left|\dot{x}_1-\dot{\tilde{\psi}}^\varepsilon_{t_1}\right|\left|\dot{x}_2-\dot{\tilde{\psi}}^\varepsilon_{t_2}\right|p^\varepsilon_{t_1,\,t_2}\left(\tilde{\psi}^\varepsilon_{t_1},\dot{x}_1,\tilde{\psi}^\varepsilon_{t_2},\dot{x}_2\right)d\dot{x}_1d\dot{x}_2\ ,$$

where $p^\varepsilon_{t_1,\,t_2}(x_1,\dot{x}_1,x_2,\dot{x}_2)$ is the joint density of the vector $(\tilde{X}^\varepsilon_{t_1},\dot{\tilde{X}}^\varepsilon_{t_1},\tilde{X}^\varepsilon_{t_2},\dot{\tilde{X}}^\varepsilon_{t_2})$.
The formula holds whether $M_2^{\varepsilon,\,\psi}$ is finite or not.
It can also be expressed as

$$M_2^{\varepsilon,\,\psi}=2\int_0^t\int_{t_1}^t I_\varepsilon(t_1,t_2)dt_2dt_1=2\int_0^t\int_0^{t-t_1}I_\varepsilon(t_1,t_1+\tau)d\tau dt_1,$$

where

$$\begin{aligned}I_\varepsilon(t_1,t_2)\ &:=\ \int_{I\!R^2}|\dot{x}_1-\dot{\tilde{\psi}}^\varepsilon_{t_1}||\dot{x}_2-\dot{\tilde{\psi}}^\varepsilon_{t_2}|p^\varepsilon_{t_1,\,t_2}(\tilde{\psi}^\varepsilon_{t_1},\dot{x}_1,\tilde{\psi}^\varepsilon_{t_2},\dot{x}_2)d\dot{x}_1d\dot{x}_2\\ &=\ p^\varepsilon_{t_1,\,t_2}(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_2})\mathbb{E}\left[|\dot{\tilde{X}}^\varepsilon_{t_1}-\dot{\tilde{\psi}}^\varepsilon_{t_1}||\,\dot{\tilde{X}}^\varepsilon_{t_2}-\dot{\tilde{\psi}}^\varepsilon_{t_2}|\,\middle|\,\tilde{X}^\varepsilon_{t_1}=\tilde{\psi}^\varepsilon_{t_1}\,,\tilde{X}^\varepsilon_{t_2}=\tilde{\psi}^\varepsilon_{t_2}\right],\end{aligned}$$

where $p^\varepsilon_{t_1,\,t_2}(x_1,x_2)$ is the joint density of the Gaussian vector $(\tilde{X}^\varepsilon_{t_1},\tilde{X}^\varepsilon_{t_2})$.
By using the following regression model:

$$(R)\qquad\begin{cases}\dot{\tilde{X}}^\varepsilon_{t_1}=\zeta_\varepsilon+\alpha_1\tilde{X}^\varepsilon_{t_1}+\alpha_2\tilde{X}^\varepsilon_{t_2}\\ \dot{\tilde{X}}^\varepsilon_{t_2}=\zeta^*_\varepsilon-\beta_1\tilde{X}^\varepsilon_{t_1}-\beta_2\tilde{X}^\varepsilon_{t_2}\end{cases}$$

where $\alpha_1(t_2-t_1)=\beta_2(t_2-t_1)=\dfrac{\rho'_\varepsilon(t_2-t_1)\rho_\varepsilon(t_2-t_1)}{1-\rho^2_\varepsilon(t_2-t_1)}$, $\alpha_2(t_2-t_1)=\beta_1(t_2-t_1)=-\dfrac{\rho'_\varepsilon(t_2-t_1)}{1-\rho^2_\varepsilon(t_2-t_1)}$,
and $(\zeta_\varepsilon,\zeta^*_\varepsilon)$ is jointly Gaussian such that $\sigma^2_\varepsilon(t_2-t_1)=Var(\zeta_\varepsilon)=Var(\zeta^*_\varepsilon)=-\rho''_\varepsilon(0)-\dfrac{(\rho'_\varepsilon(t_2-t_1))^2}{1-\rho^2_\varepsilon(t_2-t_1)}$
and $Cov(\zeta_\varepsilon,\zeta^*_\varepsilon)=-\rho''_\varepsilon(t_2-t_1)-\dfrac{(\rho'_\varepsilon(t_2-t_1))^2\rho_\varepsilon(t_2-t_1)}{1-\rho^2_\varepsilon(t_2-t_1)}$,

we obtain

$$I_\varepsilon(t_1,t_1+\tau) = p_\tau^\varepsilon(\tilde\psi^\varepsilon_{t_1},\tilde\psi^\varepsilon_{t_1+\tau})\mathbb{E}\left[|\zeta_\varepsilon+\alpha_1(\tau)\tilde\psi^\varepsilon_{t_1}+\alpha_2(\tau)\tilde\psi^\varepsilon_{t_1+\tau}-\dot{\tilde\psi}^\varepsilon_{t_1}||\zeta^*_\varepsilon-\alpha_2(\tau)\tilde\psi^\varepsilon_{t_1}-\alpha_1(\tau)\tilde\psi^\varepsilon_{t_1+\tau}-\dot{\tilde\psi}^\varepsilon_{t_1+\tau}|\right].$$

For each $\tau>0$ and by using the uniform convergence, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon(t_1,t_1+\tau) = I(t_1,t_1+\tau) := \int_{\mathbb{R}^2}|\dot x_1-\dot\psi_{t_1}||\dot x_2-\dot\psi_{t_1+\tau}|p_{t_1,t_1+\tau}(\psi_{t_1},\dot x_1,\psi_{t_1+\tau},\dot x_2)d\dot x_1 d\dot x_2,$$

and thus $\lim_{\varepsilon\to 0}\int_0^t\int_\delta^t I_\varepsilon(t_1,t_1+\tau)d\tau dt_1 = \int_0^t\int_\delta^t I(t_1,t_1+\tau)d\tau dt_1$.
We now have to prove that

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_0^t\int_0^\delta I_\varepsilon(t_1,t_1+\tau)\,d\tau dt_1 = 0.$$

We can write

$$\int_0^t\int_0^\delta I_\varepsilon(t_1,t_1+\tau)\,d\tau dt_1 \le \sum_{i=1}^3\sum_{j=1}^4\int_0^t\int_0^\delta p_\tau^\varepsilon(\tilde\psi^\varepsilon_{t_1},\tilde\psi^\varepsilon_{t_1+\tau})\mathbb{E}[I_iJ_j]d\tau dt_1,$$

where

$$I_1 = \left|\zeta_\varepsilon - \frac{\rho'_\varepsilon(\tau)}{1+\rho_\varepsilon(\tau)}\tilde\psi^\varepsilon_{t_1}\right| \quad , \quad I_2 = \left|\left(1+\frac{\tau\rho'_\varepsilon(\tau)}{1-\rho^2_\varepsilon(\tau)}\right)\dot{\tilde\psi}^\varepsilon_{t_1+\eta\tau}\right| \quad , \quad I_3 = |\dot{\tilde\psi}^\varepsilon_{t_1+\eta\tau}-\dot{\tilde\psi}^\varepsilon_{t_1}|$$

and

$$J_1 = \left|\zeta^*_\varepsilon + \frac{\rho'_\varepsilon(\tau)}{1+\rho_\varepsilon(\tau)}\tilde\psi^\varepsilon_{t_1}\right|, \qquad J_2 = \left|\rho_\varepsilon(\tau)\left(1+\tfrac{\tau\rho'_\varepsilon(\tau)}{1-\rho^2_\varepsilon(\tau)}\right)\dot{\tilde\psi}^\varepsilon_{t_1+\nu\tau}\right|,$$
$$J_3 = |(\rho_\varepsilon(\tau)-1)\dot{\tilde\psi}^\varepsilon_{t_1+\nu\tau}|, \qquad J_4 = |\dot{\tilde\psi}^\varepsilon_{t_1+\nu\tau}-\dot{\tilde\psi}^\varepsilon_{t_1+\tau}|,$$

with $0\le\ \eta\ ,\ \nu\ \le 1$.
Let C be some positive constant which may vary from equation to equation.
First we have

$$\int_0^t\int_0^\delta p_\tau^\varepsilon(\psi^\varepsilon_{t_1},\psi^\varepsilon_{t_1+\tau})\mathbb{E}[I_1J_1]d\tau dt_1 \le\ C\int_0^\delta f_\varepsilon(\tau)\frac{1}{\tau}\left(\sigma^2_\varepsilon(\tau)+\left(\frac{\rho'_\varepsilon(\tau)}{1+\rho_\varepsilon(\tau)}\right)^2\right)d\tau, \tag{36}$$

where $f_\varepsilon(\tau) := \dfrac{\tau}{\sqrt{1-\rho_\varepsilon(\tau)}}$.
Note that

$$f_\varepsilon(\tau) = \left(\frac{-\rho''_\varepsilon(0)}{2}-\frac{\theta_\varepsilon(\tau)}{\tau^2}\right)^{-1/2} \underset{\varepsilon\to 0}{\longrightarrow} \frac{\tau}{\sqrt{1-r(\tau)}}.$$

By similar techniques as the ones introduced to prove the lemmas in [15], it can be shown that
f_ε is uniformly bounded when $\varepsilon\to 0$ in $[0,\delta]$, $\delta>0$,

$$f_\varepsilon(\tau)\ \le\ C, \tag{37}$$

that

$$\frac{\sigma^2_\varepsilon(\tau)}{\tau}\ \le\ C\ \frac{\tau\theta'_\varepsilon(\tau)-\theta_\varepsilon(\tau)}{\tau^3}\ \le\ C\ L(\tau),$$

therefore, under the Geman condition, by applying (37) and the Dominated Convergence Theorem, that

$$\lim_{\varepsilon\to 0}\int_0^\delta\frac{\sigma^2_\varepsilon(\tau)}{\sqrt{1-\rho_\varepsilon(\tau)}}d\tau\ =\ \int_0^\delta\frac{\sigma^2(\tau)}{\sqrt{1-r(\tau)}}d\tau, \quad \text{with} \quad \sigma^2(\tau) := -r''(0)-\frac{r'^{\,2}(\tau)}{1-r^2(\tau)}, \tag{38}$$

which tends to zero as $\delta \to 0$.
It can also be proved that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^\delta \frac{1}{\sqrt{1-\rho_\varepsilon(\tau)}}\left(\frac{\rho'_\varepsilon(\tau)}{1+\rho_\varepsilon(\tau)}\right)^2 d\tau = 0, \tag{39}$$

by using the uniform bound (37) for $f_\varepsilon(\tau)$, and the fact that $\tau\left(\dfrac{\rho'_\varepsilon(\tau)}{\tau}\right)^2 \leq C\tau(r''(0))^2$.
We can conclude by combining (36), (37), (38) and (39).
Now let us study $\displaystyle\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_1J_2]d\tau dt_1$.
We have

$$\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_1J_2]d\tau dt_1 \leq C\left\{\int_0^\delta f_\varepsilon(\tau)d\tau \;+\; \int_0^\delta (f_\varepsilon(\tau))^{3/2}\left(\frac{\theta''(\tau)}{\tau}+\frac{\theta(\tau)}{\tau^3}\right)d\tau\right\}.$$

Again, the uniform bound (37) for $f_\varepsilon(\tau)$ and the Geman condition allow to conclude that $\displaystyle\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_1J_2]d\tau dt_1 = 0$.
The term $\displaystyle\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_2J_1]d\tau dt_1$ can be bounded in the same way.
Now, since $p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\; J_3 \;\leq\; C(\sqrt{1-\rho_\varepsilon(\tau)}\,)$, we obtain

$$\sum_{i=1}^3 \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_iJ_3]d\tau dt_1 = 0.$$

Finally, the fact that $|\dot{\tilde{\psi}}^\varepsilon_{t+\tau}-\dot{\tilde{\psi}}^\varepsilon_t| \leq\; C\gamma(\tau)$ and $p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})I_3 \leq\; C\; f_\varepsilon(\tau)\dfrac{\gamma(\tau)}{\tau}$ implies

$$\sum_{i=1}^3 \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^t\int_0^\delta p^\varepsilon_\tau(\tilde{\psi}^\varepsilon_{t_1},\tilde{\psi}^\varepsilon_{t_1+\tau})\mathbb{E}[I_iJ_4]d\tau dt_1 = 0.$$

Hence the result (34).

• *Conclusion.*
Since, by Fatou lemma, for each Q positive,

$$\begin{aligned}\sum_{q=0}^Q \;\mathbb{E}\;&\left[\left(\int_0^t\sum_{l=0}^q \frac{H_{q-l}(\psi_s)\varphi(\psi_s)}{(q-l)!}\,a_l\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right)H_{q-l}(X_s)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds\right)^2\right]\\ &\leq\; \lim_{\varepsilon\to 0}\mathbb{E}\left[\left(\sqrt{\frac{-r''_\varepsilon(0)}{r_\varepsilon(0)}}\,N^\psi_{\varepsilon,\,t}\right)^2\right] = \mathbb{E}[(\sqrt{-r''(0)}\,N^X_t(\psi))^2],\end{aligned}$$

the formal expansion in the right hand side of (5) defines a random variable in $L^2(\Omega)$, denoted by $\mathcal{N}_t$.
Moreover

$$\mathbb{E}\left[\left(\frac{1}{\eta}N^X_t(\psi)-\mathcal{N}_t\right)^2\right] \;\leq\; 2\left(\mathbb{E}\left[\frac{N^X_t(\psi)}{\sqrt{-r''(0)}}-\sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}}\,N^\psi_{\varepsilon,\,t}\right]^2 \;+\; \mathbb{E}\left[\sqrt{\frac{r_\varepsilon(0)}{-r''_\varepsilon(0)}}\,N^\psi_{\varepsilon,\,t}-\mathcal{N}_t\right]^2\right).$$

The first term of the right hand side, tends to zero as shown in the previous step. For the second term, an argument of continuity for the projections into the chaos entails the result (see [12]).

Note that we considered the case where $r(0) = 1$; it can be easily generalized to a process X satisfying $\mathbb{E}[X_t^2] = r(0)$ when noticing that $N_t^X(\psi) = N^Y_{t/\sqrt{r(0)}}(0)$, where $Y_s = \frac{1}{\sqrt{r(0)}}\left(X\left(s\sqrt{r(0)}\right) - \psi\left(s\sqrt{r(0)}\right)\right)$.
□

A2 - Proof of Theorems 1 and 2

1) Periodic curve

We will consider the case $t = np$, where p is the curve's periodicity.
It will be straightforward to deduce the general case.

We can write

$$N_t^X(\psi) - \mathbb{E}[N_t^X(\psi)] = \sqrt{\frac{-r''(0)}{r(0)}}\sum_{q=1}^{\infty}\int_0^t \sum_{l=0}^{q}\kappa_{ql}(s)H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds.$$

Let define

$$\begin{aligned} Y_Q^k &:= \sum_{q=Q+1}^{\infty}\int_{(k-1)p}^{kp}\sum_{l=0}^{q}\kappa_{ql}(s)H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds, \\ &= \sum_{q=Q+1}^{\infty}\int_0^p\sum_{l=0}^{q}\kappa_{ql}(u)H_{q-l}\left(\frac{X_{u+(k-1)p}}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_{u+(k-1)p}}{\sqrt{-r''(0)}}\right)du \quad \text{(by periodicity)}. \end{aligned}$$

We have

$$\sum_{k=1}^{n} Y_Q^k = \sum_{q=Q+1}^{\infty}\int_0^{np}\sum_{l=0}^{q}\kappa_{ql}(s)H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds.$$

(i) Let us prove that

$$\lim_{Q\to\infty}\lim_{n\to\infty}\frac{1}{n}\,\mathbb{E}\left[\left(\sum_{k=1}^{n}Y_Q^k\right)^2\right] = 0. \tag{40}$$

We have

$$\mathbb{E}[(Y_Q^k)^2] = \sum_{q=Q+1}^{\infty}\mathbb{E}\left(\left[\int_0^p\sum_{l=0}^{q}\kappa_{ql}(u)H_{q-l}\left(\frac{X_u}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_u}{\sqrt{-r''(0)}}\right)du\right]^2\right) = \mathbb{E}[(Y_Q^1)^2],$$

by using the orthogonality of the chaos, the stationarity and the p-periodicity.
Therefore we obtain

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[(Y_Q^k)^2] = \mathbb{E}[(Y_Q^1)^2] \;\to\; 0 \quad \text{as } Q\to\infty. \tag{41}$$

Suppose now that $k > m$ and write

$$\mathbb{E}\left[Y_Q^k Y_Q^m\right] = \sum_{q=Q+1}^{\infty}\int_0^p\int_0^p\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(v)I_{ql_1l_2}(k-m,u,v)dudv,$$

where $I_{ql_1l_2}(k-m,u,v)$ is defined for $k > m$ by

$$I_{ql_1l_2}(k-m,u,v) = \mathbb{E}\left[H_{q-l_1}\left(\frac{X_{u+(k-m)p}}{\sqrt{r(0)}}\right)H_{l_1}\left(\frac{\dot{X}_{u+(k-m)p}}{\sqrt{-r''(0)}}\right)H_{q-l_2}\left(\frac{X_v}{\sqrt{r(0)}}\right)H_{l_2}\left(\frac{\dot{X}_v}{\sqrt{-r''(0)}}\right)\right]. \tag{42}$$

Thus we have

$$\sum_{k\neq m} \mathbb{E}[Y_Q^k Y_Q^m] = 2\sum_{k>m}\sum_{q=Q+1}^{\infty}\int_0^p\int_0^p \sum_{l_1=0}^{q}\sum_{l_2=0}^{q} \kappa_{ql_1}(u)\kappa_{ql_2}(v) I_{ql_1l_2}(k-m,u,v)dudv$$
$$= 2n\sum_{j=1}^{n-1}(1-\frac{j}{n})\sum_{q=Q+1}^{\infty}\int_0^p\int_0^p \sum_{l_1=0}^{q}\sum_{l_2=0}^{q} \kappa_{ql_1}(u)\kappa_{ql_2}(v) I_{ql_1l_2}(j,u,v)dudv. \qquad (43)$$

By hypothesis we have $\int_0^\infty \chi(u)du < \infty$. The continuity of the functions defining χ implies that there exists a positive integer a such that $\chi(v) < \rho$ for $v > a$.
So (43) can be rewritten as

$$\sum_{k\neq m} \mathbb{E}[Y_Q^k Y_Q^m] = 2(I_{a1}+I_{a2}), \qquad (44)$$

$$\text{with} \quad I_{a1} := \sum_{j=1}^{a}\left(1-\frac{j}{n}\right)\sum_{q=Q+1}^{\infty}\int_0^p\int_0^p \sum_{l_1=0}^{q}\sum_{l_2=0}^{q} \kappa_{ql_1}(u)\kappa_{ql_2}(v) I_{ql_1l_2}(j,u,v)dudv$$

$$\text{and} \quad I_{a2} := \sum_{j=a+1}^{n-1}\left(1-\frac{j}{n}\right)\sum_{q=Q+1}^{\infty}\int_0^p\int_0^p \sum_{l_1=0}^{q}\sum_{l_2=0}^{q} \kappa_{ql_1}(u)\kappa_{ql_2}(v) I_{ql_1l_2}(j,u,v)dudv.$$

On one hand we have

$$I_{a1} \leq \sum_{j=1}^{a}\left(1-\frac{j}{n}\right)\sum_{q=Q+1}^{\infty}\mathbb{E}\left(\left[\int_0^p \sum_{l_1=0}^{q}\kappa_{ql_1}(u) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right) du\right]^2\right) \underset{Q\to\infty}{\to} 0. \qquad (45)$$

On the other hand, I_{a2} can be bounded as

$$\begin{aligned} I_{a2} \leq{}& \sum_{j=a+1}^{n-1}\left(1-\frac{j}{n}\right)\sum_{q=Q+1}^{\infty}\int_0^p\int_0^p \chi^q(u+jp-v)\ \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(u) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) \\ &\times \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(v) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) dvdu \\ \leq{}& \sum_{q=Q+1}^{\infty}\int_0^p\int_{u+ap}^{u+np} \chi^q(w)\ \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(u) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) \\ &\times \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(u-w) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) dwdu \\ \leq{}& \sum_{q=Q+1}^{\infty}\rho^{q-2}\int_{ap}^{\infty}\chi^2(w)\int_0^p \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(u) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) \\ &\times \mathbb{E}^{1/2}\left(\left[\sum_{l_1=0}^{q}\kappa_{ql_1}(u-w) H_{q-l_1}\left(\frac{X_0}{\sqrt{r(0)}}\right) H_{l_1}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]^2\right) dudw, \end{aligned}$$

by using Arcones inequality (see [1] or [23]), then the change of variables $w = u + jp - v$ and finally the periodicity of κ_{ql}.

This last upper bound tends to 0 as $Q \to \infty$ since

$$\begin{aligned}
\mathbb{E}\left(\left[\sum_{l=0}^{q}\kappa_{ql}(u)H_{q-l}(\frac{X_0}{\sqrt{r(0)}})H_l(\frac{\dot{X}_0}{\sqrt{-r''(0)}})\right]^2\right) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\sum_{l=0}^{q}\kappa_{ql}(u)H_{q-l}(x_1)H_l(x_2)\right)^2 e^{-\frac{x_1^2+x_2^2}{2}}\,dx_1dx_2 \\
&= \kappa_{ql}^2(u)\ (q-l)!\ l! \\
&= \sum_{l=0}^{q}\frac{H_{q-l}^2\left(\frac{\psi_s}{\sqrt{r(0)}}\right)\varphi^2\left(\frac{\psi_s}{\sqrt{r(0)}}\right)}{(q-l)!}\,a_l^2\left(\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right)l! \\
&\le C\,\mathbb{E}\left[\left(Z+\frac{\dot{\psi}_s}{\sqrt{-r''(0)}}\right)^2\right] \ \le\ 2C\left(1+\frac{||\dot{\psi}||_\infty^2}{-r''(0)}\right),
\end{aligned}$$

with C some constant independent of q, as a consequence of Proposition 3 in Imkeller *et al.* (see [11]). Hence $I_{a2} \underset{Q\to\infty}{\longrightarrow} 0$, which combined with (45), (44) and (41) provide (40).

(ii) We are now interested in the asymptotical variance of $\mathcal{F}_n := \dfrac{N_{np}^X(\psi)-\mathbb{E}[N_{np}^X(\psi)]}{\sqrt{n}}$.

We have

$$\sigma_n^2 \ := \ var(\mathcal{F}_n) \ = \ \frac{-r''(0)}{r(0)}\sum_{q=1}^{\infty}\sigma_n^2(q), \quad \text{with}$$

$$\sigma_n^2(q) := \frac{1}{n}\mathbb{E}\left(\left[\int_0^{np}\sum_{l=0}^{q}\kappa_{ql}(s)H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right)ds\right]^2\right).$$

Note that the Fatou lemma implies that

$$\sum_{q=2}^{\infty}\lim_{n\to\infty}\sigma_n^2(q) \ \le \ \lim_{n\to\infty}\ \frac{1}{n}\ \mathbb{E}\left[\left(\sum_{k=1}^{n}Y_1^k\right)^2\right] \ = \ \mathbb{E}[(Y_1^1)^2] < \infty.$$

Let us study the asymptotical behavior as $n \to \infty$ of each component $\sigma_n^2(q)$.
We can write

$$\begin{aligned}
\sigma_n^2(q) &= \frac{1}{n}\ \mathbb{E}\left[\left(\sum_{k=1}^{n}\int_0^p\sum_{l=0}^{q}\kappa_{ql}(u)H_{q-l}\left(\frac{X_{u+(k-1)p}}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_{u+(k-1)p}}{\sqrt{-r''(0)}}\right)du\right)^2\right] \\
&:= \frac{1}{n}\ \mathbb{E}\left[\left(\sum_{k=1}^{n}Z_k^q\right)^2\right] = \mathbb{E}\left[(Z_1^q)^2\right] + \frac{2}{n}\sum_{k_1=1}^{n-1}\sum_{k_2=k_1+1}^{n}\mathbb{E}[Z_{k_1}^q Z_{k_2}^q].
\end{aligned}$$

But

$\mathbb{E}\left[(Z_1^q)^2\right]$

$$= 2\int_0^p\int_0^u\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(u-z)\mathbb{E}\left[H_{q-l_1}\left(\frac{X_z}{\sqrt{r(0)}}\right)H_{l_1}\left(\frac{\dot{X}_z}{\sqrt{-r''(0)}}\right)H_{q-l_2}\left(\frac{X_0}{\sqrt{r(0)}}\right)H_{l_2}\left(\frac{\dot{X}_0}{\sqrt{-r''(0)}}\right)\right]dzdv,$$

and, by stationarity, $\mathbb{E}[Z_{k_1}^q Z_{k_2}^q] \ = \ \displaystyle\int_0^p\int_0^p\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(v)I_{ql_1l_2}(k_2-k_1,u,v)dudv,$

$I_{ql_1l_2}$ being defined in (42),

thus it comes

$$\begin{aligned}\frac{1}{n}\sum_{k_1=1}^{n-1}\sum_{k_2=k_1+1}^{n}\mathbb{E}[Z^q_{k_1}Z^q_{k_2}] &= \sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\int_0^p\int_0^p\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(v)I_{ql_1l_2}(j,u-v,0)dudv\\ &= \int_0^p\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\int_{u+(j-1)p}^{u+jp}\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(u-z)I_{ql_1l_2}(0,z,0)dzdu\\ &= \int_0^p\int_u^\infty\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\mathbb{1}_{[u+(j-1)p,u+jp]}(z)\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(u-z)I_{ql_1l_2}(0,z,0)dzdu,\end{aligned}$$

with the change of variables $z=u-v+jp$, and using the periodicity and stationarity of κ_{ql}.
Notice that $\displaystyle\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\mathbb{1}_{[u+(j-1)p,u+jp]}(z)\underset{n\to\infty}{\to}\mathbb{1}_{[u,\infty)}(z)$.
On the other hand, we can apply the Dominated Convergence Theorem, splitting the inner integral into two parts, namely on the interval $[u,a]$, with a chosen such that $\chi(z)<\rho<1$ for $z>a$, and on $[a,\infty]$ respectively.
Finally we obtain

$$\frac{1}{n}\sum_{k_1\neq k_2}\mathbb{E}[Z^q_{k_1}Z^q_{k_2}]\underset{n\to\infty}{\to}2\int_0^p\int_u^\infty\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\kappa_{ql_1}(u)\kappa_{ql_2}(u-z)\ I_{ql_1l_2}(0,z,0)\ dzdu,$$

hence

$$\sigma_n^2(q)\underset{n\to\infty}{\to}2\int_0^\infty\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\left(\int_0^p\kappa_{ql_1}(u)\kappa_{ql_2}(u-z)du\right)\ I_{ql_1l_2}(0,z,0)\ dz,$$

namely

$$\sigma^2(q):=\lim_{n\to\infty}\sigma_n^2(q)\ =\ 2\int_0^\infty\sum_{l_1=0}^{q}\sum_{l_2=0}^{q}\tilde{\kappa}_{ql_1}*\kappa_{ql_2}(z)\ I_{ql_1l_2}(0,z,0)\ dz,\tag{46}$$

with the notation $\tilde{h}(u):=h(-u)$ and $h*g(u):=\displaystyle\int_0^p h(u-v)g(v)dv$.
As in [14] (p.653), we can conclude that

$$\mathbb{E}\left[\mathcal{F}_n^2\right]\ :=\ \mathbb{E}\left[\left(\frac{N^X_{np}(\psi)-\mathbb{E}N^X_{np}(\psi)}{\sqrt{n}}\right)^2\right]\underset{n\to\infty}{\longrightarrow}\frac{-r''(0)}{r(0)}\sum_{q=1}^{\infty}\sigma^2(q),\quad \sigma^2(q)\text{ satisfying (46).}$$

(iii) CLT for $\mathcal{F}_n$ under the hypothesis of m-dependence on X.
Let us define

$$\begin{aligned}\mathcal{F}_{Q,n}\ &:=\ \frac{1}{\sqrt{n}}\int_0^{np}g_Q(s,X_s,\dot{X}_s)\ ds,\\ \text{with}\quad g_Q(s,X_s,\dot{X}_s)\ &:=\ \sqrt{\frac{-r''(0)}{r(0)}}\sum_{q=0}^{Q}\sum_{l=0}^{q}\kappa_{ql}(s)H_{q-l}\left(\frac{X_s}{\sqrt{r(0)}}\right)H_l\left(\frac{\dot{X}_s}{\sqrt{-r''(0)}}\right).\end{aligned}$$

By using (40), it holds that

$$\lim_{Q\to\infty}\lim_{n\to\infty}\mathbb{E}\left[(\mathcal{F}_n-\mathcal{F}_{Q,n})^2\right]=0.$$

As pointed out in [14], we just have to show that $\mathcal{F}_{Q,n}$ satisfies a CLT to obtain the same result for $\mathcal{F}_n$.
Since can write

$$\mathcal{F}_{Q,n}\ =\ \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\int_0^p g_Q(s,X_{s+(k-1)p},\dot{X}_{s+(k-1)p})ds\ :=\ \frac{1}{\sqrt{n}}\sum_{k=1}^{n}Y_k=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\theta_{(k-1)p}Y_1,$$

where θ_t denotes the shift operator of path t associated to X and where the random variables Y_k are, under the m-dependence condition on X, $(\frac{m}{p}+1)$-dependent and identically distributed,
then an immediate application of the Hoeffding-Robbins CLT for m-dependent sequences (see [10]) provides

$$\mathcal{F}_{Q,n} \underset{n\to\infty}{\xrightarrow{d}} \mathcal{N}(0, \sum_{q=1}^{Q} \sigma_m^2(q)), \quad \text{with } \sigma_m^2(q) = 2\int_0^m \sum_{l_1=0}^{q}\sum_{l_2=0}^{q} \tilde{\kappa}_{ql_1} * \kappa_{ql_2}(z)\ I_{ql_1l_2}(0,z,0)\ dz,$$

with the notation (42).

(iv) Generalization.
The result can be extended now under the more general weak dependence assumption (8), aproaching in $L^2(\Omega)$ the process X by a m-dependent process, as $t\to\infty$ (see [14], Lemma 3). □

2) Linear curve and specular points

Since the correlation function of $\partial_x W(0,x)$ is $-r''(x)$, Proposition 1 provides the Hermite expansion of $N_{[0,x]}$ under the Geman condition (13), from which the expectation is deduced:

$$\mathbb{E}[N_{[0,x]}] \;=\; \frac{\gamma}{\eta}\int_0^x \varphi\left(\frac{\kappa s}{\eta}\right) a_0\left(\frac{-\kappa}{\gamma}\right) ds, \text{ i.e. (15)},$$

as well as the variance.
Let us study the limit of the variance of $N_{[0,\,x]}$ as $x\to\infty$, under the m-dependence assumption on $\partial_x W(0,x)$.
As a consequence of the diagram formula (see [5]) and the Dominated Convergence Theorem , we obtain for each $q\geq 1$

$$\lim_{x\to\infty}\int_0^x\int_0^{x-s} \mathbb{E}\left[F_q\left(s,\frac{\partial_x W(0,s)}{\eta},\frac{\partial_{xx}W(0,s)}{\gamma}\right) F_q\left(s+\tau,\frac{\partial_x W(0,s+\tau)}{\eta},\frac{\partial_{xx}W(0,s+\tau)}{\gamma}\right)\right] d\tau ds$$
$$=\int_0^\infty\int_0^{m} \mathbb{E}\left[F_q\left(s,\frac{\partial_x W(0,s)}{\eta},\frac{\partial_{xx}W(0,s)}{\gamma}\right) F_q\left(s+\tau,\frac{\partial_x W(0,s+\tau)}{\eta},\frac{\partial_{xx}W(0,s+\tau)}{\gamma}\right)\right] d\tau ds,$$

F_q being defined in Theorem 2.
We shall study the asymptotic behavior as $x\to\infty$ of the second moment of $N_{[0,x]}$.
First let us prove that the second factorial moment $M_2^{\psi}(x)$ of $N_{[0,x]}$ is finite as $x\to\infty$.
For ease of notation we shall suppose bellow that $-r''(0)=1$.
We have $\quad M_2^{\psi}(x) \;=\; 2\displaystyle\int_0^x\int_0^{x-s} p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)d\tau ds$, where
$G(s,\tau)=\mathbb{E}\left(\left|\zeta-\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\kappa s-\frac{r'''(\tau)}{1-(r''(\tau))^2}\kappa(s+\tau)+\kappa\right|\left|\zeta^*+\frac{r'''(\tau)}{1-(r''(\tau))^2}\kappa s+\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\kappa(s+\tau)+\kappa\right|\right)$,
$\psi(x)=-\kappa x$, p_τ is the density of $(\partial_x W(0,s),\partial_x W(0,s+\tau))$ and (ζ,ζ^*) is defined in the same way as in (R) given in the proof of Proposition 1. Then

$$M_2^{\psi}(x) \;=\; 2\int_0^x\int_\delta^{x-s} p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau + 2\int_0^x\int_0^\delta p_\tau(-\kappa s,-\kappa(s+\tau))G(s,\tau)dsd\tau.$$

By using the Cauchy-Schwarz inequality, we have

$$G(s,\tau)\leq\ C\left[\mathbb{E}\zeta^2+\left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\right)^2+\left(\frac{r'''(\tau)r''(\tau)}{1-(r''(\tau))^2}\right)^2+\kappa^2\right](s^2+(s+\tau)^2).$$

Moreover, we can write

$$p_\tau(-\kappa s,-\kappa(s+\tau)) \;=\; \frac{1}{2\pi\sqrt{1-(r''(\tau))^2}}\, e^{-\frac{\kappa^2}{1-r''(\tau)}\left(s+\frac{1}{2}\tau\right)^2} e^{-\frac{\kappa^2\tau^2}{4(1+r''(\tau))}},$$

providing $\quad p_\tau(-\kappa s, -\kappa(s+\tau)) \leq \; C\, e^{-\frac{\kappa^2}{1-r''(\tau)}(s+\frac{1}{2}\tau)^2}$, if $\tau \in [\delta, x-s]$.
Thus applying the Dominated Convergence Theorem provides

$$\lim_{x\to\infty} \int_0^x \int_\delta^{x-s} p_\tau(-\kappa s, -\kappa(s+\tau))G(s,\tau)dsd\tau = \int_0^\infty \int_\delta^m p_\tau(-\kappa s, -\kappa(s+\tau))G(s,\tau)dsd\tau.$$

When working in the interval $[0,\delta]$, we proceed as in the proof of Proposition 1 when bounding the terms $\mathbb{E}[I_i J_i]$, to bound all terms defining $G(s,\tau)$ by functions belonging to $L^2[0,\delta]$ in order to be able to apply the Dominated Convergence Theorem.
The we can conclude that $\lim\limits_{x\to\infty} M_2^\psi(x) \; = \; 2\int_0^\infty \int_0^m p_\tau(-\kappa s, -\kappa(s+\tau))G(s,\tau)dsd\tau < \infty.$
Now applying the Fatou lemma gives

$$\begin{aligned} &\infty > \lim_{x\to\infty} \frac{\eta}{\gamma}\mathbb{E}(N^2_{[0,x]}) \geq \lim_{x\to\infty} Var\left(\frac{\eta}{\gamma}\, N_{[0,x]}\right) \\ \geq \quad & 2\sum_{q=1}^\infty \int_0^\infty \int_0^m \mathbb{E}\left[F_q\left(s, \frac{\partial_x W(0,s)}{\eta}, \frac{\partial_{xx} W(0,s)}{\gamma}\right) F_q\left(s+\tau, \frac{\partial_x W(0,s+\tau)}{\eta}, \frac{\partial_{xx} W(0,s+\tau)}{\gamma}\right)\right] dsd\tau. \end{aligned}$$

But $N_{[0,x]} \uparrow N_{[0,\,\infty]}$ and so $\mathbb{E}(N^2_{[0,x]}) \;\to\; \mathbb{E}(N^2_{[0,\,\infty]})$, as $x\to\infty$.
We can then conclude to (16) of Theorem 2. $\Box$

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